

Exercise 4

Use the *Laplace transform method* to solve the Volterra integral equations:

$$u(x) = 1 + 3 \int_0^x (x-t)u(t) dt$$

Solution

The Laplace transform of a function $f(x)$ is defined as

$$\mathcal{L}\{f(x)\} = F(s) = \int_0^{\infty} e^{-sx} f(x) dx.$$

According to the convolution theorem, the product of two Laplace transforms can be expressed as a transformed convolution integral.

$$F(s)G(s) = \mathcal{L} \left\{ \int_0^x f(x-t)g(t) dt \right\}$$

Take the Laplace transform of both sides of the integral equation.

$$\begin{aligned} \mathcal{L}\{u(x)\} &= \mathcal{L} \left\{ 1 + 3 \int_0^x (x-t)u(t) dt \right\} \\ U(s) &= \mathcal{L}\{1\} + 3\mathcal{L} \left\{ \int_0^x (x-t)u(t) dt \right\} \\ &= \mathcal{L}\{1\} + 3\mathcal{L}\{x\}U(s) \\ &= \frac{1}{s} + 3 \left(\frac{1}{s^2} \right) U(s) \\ &= \frac{1}{s} + \frac{3}{s^2} U(s) \end{aligned}$$

Solve for $U(s)$.

$$\begin{aligned} \left(1 - \frac{3}{s^2} \right) U(s) &= \frac{1}{s} \\ U(s) &= \frac{\frac{1}{s}}{1 - \frac{3}{s^2}} \\ &= \frac{s}{s^2 - 3} \end{aligned}$$

Take the inverse Laplace transform of $U(s)$ to get the desired solution.

$$\begin{aligned} u(x) &= \mathcal{L}^{-1}\{U(s)\} \\ &= \mathcal{L}^{-1} \left\{ \frac{s}{s^2 - 3} \right\} \\ &= \cosh(\sqrt{3}x) \end{aligned}$$